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A New Estimate for the Approximation of Functions by Hermite–Fejér Interpolation Polynomials

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A new estimate is derived for the error committed in approximating a continuous function by Hermite–Fejér interpolation polynomials on the Chebyshev nodes of the first kind. The estimate obtained reflects the fact that the polynomials interpolate the function which is being approximated.

1. A BRIEF HISTORY OF ESTIMATES

One of the proofs of Weierstrass' approximation theorem using interpolation polynomials was presented by Fejér [3] in 1916. We shall begin by recalling this result.

Let $x_k = \cos((2k-1)\pi/2n)$, $k = 1, 2,..., n$, denote the zeros of the Chebyshev polynomial of the first kind, $T_n(x) = \cos(n \arccos x)$, $-1 \leqslant x \leqslant 1$. If $f \in C([-1, 1])$, then there is a unique polynomial $H_{2n-1}(f, x)$ of degree $\leqslant 2n-1$ such that

$$H_{2n-1}(f, x_k) = f(x_k), \qquad k = 1, 2,..., n,$$

and

$$H'_{2n-1}(f, x_k) = 0, \qquad k = 1, 2,..., n.$$

This polynomial is known as the Hermite–Fejér interpolation polynomial based on the zeros of $T_n(x)$.

Fejér's result is the following:

THEOREM 1 (L. Fejér). *If $f \in C([-1, 1])$ then* $\lim_{n\to\infty} \|H_{2n-1}(f) - f\| = 0$, *where $\|\cdot\|$ denotes the uniform norm on the space $C([-1, 1])$.*

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The first estimate of the rate of convergence of the polynomials was derived by Popoviciu [8] in 1950. The estimate is given in terms of the modulus of continuity of f which is defined by

$$\omega(f;\delta) := \sup\{|f(x)-f(y)|: -1 \leqslant x,\ y \leqslant 1, |x-y| \leqslant \delta\}.$$

Popoviciu's result is the following:

THEOREM 2 (T. Popoviciu). *For* $n = 1, 2, 3, \ldots$, $\|H_{2n-1}(f) - f\| \leqslant 2\omega(f; n^{-1/2})$.

Bojanic [1] reports that a similar result was proved by Shisha *et al.* [11].

This estimate was improved by Moldovan [7] and, from quite a different approach, by Shisha and Mond [10]. Their results are summed up in the following:

THEOREM 3 (E. Moldovan, O. Shisha, B. Mond). *For* $n = 4, 5, 6, \ldots$, $\|H_{2n-1}(f) - f\| \leqslant C_1\omega(f; (\ln n)/n)$.

Here C_1, and later C_2, $C_3, \ldots$, are absolute positive constants.

In one sense, this is the best possible estimate. For, if $g(x) = |x|$, then one can show that

$$C_2(\ln n)/n \leqslant \|H_{2n-1}(g) - g\| \leqslant C_1(\ln n)/n = C_1\omega(g; (\ln n)/n)$$

for infinitely many values of n. Thus, the function $(\ln n)/n$ which appears in Theorem 3 cannot be replaced by a function of smaller order. In this case, Theorem 3 gives the best possible estimate, but for the function $g(x) = |x|^\alpha$ $(0 < \alpha < 1)$ the estimate $(\ln n/n)^\alpha$ given by Theorem 3 is not good, the correct estimate being of the order $1/n^\alpha$.

The next major improvement in the estimate was established by Bojanic [1], who gleaned an idea used by Steckin in a paper on Fourier series. Before stating Bojanic's result we must define a particular class of functions.

Let Ω: $[0, \infty) \to [0, \infty)$ be an increasing, subadditive, continuous function such that $\Omega(0) = 0$. Then define $C(\Omega)$ to be the following class of functions

$$C(\Omega) := \{f \in C([-1, 1]) : \omega(f; \delta) \leqslant \Omega(\delta) \text{ for all } \delta \geqslant 0\}.$$

Bojanic's result is as follows:

THEOREM 4 (R. Bojanic). *There exist positive constants* C_3, C_4 *such that, for* $n = 2, 3, 4, \ldots$,

$$\frac{C_3}{n}\sum_{k=2}^{n}\Omega(1/k) \leqslant \sup\{\|H_{2n-1}(f) - f\| : f \in C(\Omega)\} \leqslant \frac{C_4}{n}\sum_{k=1}^{n}\Omega(1/k).$$

This result is an improvement on Theorem 3. By using the properties of a modulus of continuity (see Lorentz [6, p. 43 et seq.]), we have

$$\|H_{2n-1}(f)-f\| \leqslant \frac{C_4}{n} \sum_{k=1}^{n} \omega\left(f; \frac{1}{k}\right)$$

and it is easy to see that this inequality gives correct estimates for all Lipschitz α functions $(0<\alpha \leqslant 1)$. It is also easy to see that

$$\frac{1}{n} \sum_{k=1}^{n} \omega\left(f; \frac{1}{k}\right) \leqslant C_5 \omega(f; (\ln n)/n),$$

and thus Bojanic's estimate improves all earlier results. His lower estimate shows that this theorem cannot be significantly improved if one considers all functions in the class $C(\Omega)$.

A possible improvement would be to show that

$$\|H_{2n-1}(f)-f\| \leqslant \frac{C}{n} \sum_{k=1}^{n} E_k(f),$$

where $E_k(f)$ is the best approximation to f by polynomials of degree $\leqslant k$. Jackson's [4, p. 16] famous theorem that $E_k(f) \leqslant 4\omega(f; 1/k)$ suggests this latter possible improvement. However, these problems are not the subject of this paper.

The next improvement in estimates came from Vértesi [12] and Saxena [9].

These authors studied the difference

$$|H_{2n-1}(f, x)-f(x)|, \qquad -1 \leqslant x \leqslant 1,$$

and obtained Bojanic's upper estimate as a corollary of their pointwise estimates. Their results may be written as follows:

THEOREM 5 (P. Vértesi, R. B. Saxena). *There is a positive constant C_6 such that, for $n \geqslant 2$ and $-1 \leqslant x \leqslant +1$,*

$$|H_{2n-1}(f, x)-f(x)| \leqslant \frac{C_6}{n} \sum_{k=1}^{n} \left[\omega\left(f; \frac{(1-x^2)^{1/2}}{k}\right)+\omega\left(f; \frac{1}{k^2}\right)\right].$$

Thus the approximation is considerably better at the end points than it may be at the centre of the interval. General lower bounds of this type for the difference $|H_{2n-1}(f, x)-f(x)|$ have never been published.

It is unfortunate that not one of the preceding estimates reflects the fact that if x is a node of interpolation then $|H_{2n-1}(f, x)-f(x)|=0$. Such an estimate was given by DeVore [2, p. 44]:

THEOREM 6 (R. A. DeVore). *For* $n = 1, 2, 3, \ldots,$ *and* $-1 \leqslant x \leqslant +1$,

$$|H_{2n-1}(f, x) - f(x)| \leqslant 2\omega(f; n^{-1/2} |T_n(x)|).$$

Notice that Theorem 6 implies that if $T_n(x) = 0$ then $H_{2n-1}(f, x) = f(x)$; that is, $H_{2n-1}(f, x)$ interpolates $f(x)$ at the zeros of $T_n(x)$. However, this estimate is not precise when x is not one of the nodes.

To remedy the situation we shall prove the following result.

THEOREM 7. *There are positive constants* C_9, C_{10} *such that, for* $n \geqslant 2$ *and* $-1 \leqslant x \leqslant +1$,

$$|H_{2n-1}(f, x) - f(x)| \leqslant \frac{C_9}{n} T_n(x)^2 \sum_{k=1}^{n} \left[\omega\left(f; \frac{(1-x^2)^{1/2}}{k}\right) + \omega\left(f; \frac{1}{k^2}\right)\right] + C_{10}\,\omega\left(f; \frac{|T_n(x)|}{n}\right)$$

2. PRELIMINARIES

Before proving Theorem 7 we shall state a few preliminary formulae and results.

An explicit formula for Hermite–Fejér polynomials of f will be required:

$$H_{2n-1}(f, x) = \sum_{k=1}^{n} f(x_k)\, h_k(x), \tag{2.1}$$

where

$$x_k = \cos((2k-1)\pi/2n),$$

$$h_k(x) = \frac{(1 - x x_k)\, T_n(x)^2}{n^2 (x - x_k)^2},$$

and

$$T_n(x) = \cos(n \arccos x).$$

It is well known that, for all $x \in [-1, 1]$,

$$h_k(x) \geqslant 0 \tag{2.2}$$

and

$$\sum_{k=1}^{n} h_k(x) = 1. \tag{2.3}$$

For each $x \in [-1, 1]$ let x_j be the node which is nearest to x. If there are two such nodes then let x_j be either one of them.

We shall require a lemma of Kis [5, p. 30]:

LEMMA 1. *For* $-1 \leqslant x = \cos\theta \leqslant +1$,

$$|f(x_k) - f(x)| \leqslant 2\omega\left(f; \frac{\sin\theta}{n}\right) + 2\omega\left(f; \frac{1}{n^2}\right) \qquad \text{if } k = j$$

$$\leqslant 5\omega\left(f; \frac{i\sin\theta}{n}\right) + 13\omega\left(f; \frac{i^2}{n^2}\right) \qquad \text{if } i = |k - j| \geqslant 1.$$

The following elementary inequalities will be useful:

LEMMA 2. *If* $0 \leqslant \alpha, \beta \leqslant \pi$ *then*

(a) $0 \leqslant \sin\alpha \leqslant 2\sin\frac{1}{2}(\alpha + \beta)$ *and*

(b) $\sin\frac{1}{2}(\alpha + \beta) \geqslant \sin\frac{1}{2}|\alpha - \beta|$.

Finally, we shall require

LEMMA 3. *Let* $x = \cos\theta$, $x_k = \cos\theta_k$ $k = 1, 2,..., n$, *and* x_j *be the node closest to* x. *Then*

$$|\theta - \theta_j| \leqslant \frac{\pi}{2n}|\cos n\theta|.$$

Proof. Suppose that $\theta_j \leqslant \theta \leqslant (\theta_j + \theta_{j+1})/2$. Other cases may be treated similarly. Then,

$$\frac{|\cos n\theta|}{\theta - \theta_j} = \frac{|\cos n\theta - \cos n\theta_j|}{\theta - \theta_j}$$

$$\geqslant \frac{|\cos(n(\theta_j + \theta_{j+1})/2) - \cos n\theta_j|}{(\theta_j + \theta_{j+1})/2 - \theta_j}$$

$$= \frac{2n}{\pi}.$$

Therefore $|\theta - \theta_j| \leqslant \pi|\cos n\theta|/2n$.

3. Proof of Theorem 7

From (2.1), (2.2), (2.3) it follows that

$$\begin{aligned}|H_{2n-1}(f, x) - f(x)| &= \left| \sum_{k=1}^{n} (f(x_k) - f(x))\, h_k(x) \right| \\ &\leqslant \sum_{k=1}^{n} |f(x_k) - f(x)|\, h_k(x) \equiv \sum_{k=1}^{n} W_k(x), \text{ say,} \\ &= \sum_{k=1}^{j-1} W_k(x) + W_j(x) + \sum_{k=j+1}^{n} W_k(x) \\ &= I_1 + I_2 + I_3, \text{ say.}\end{aligned}$$

We shall proceed to estimate each of these three terms. Clearly if $j = 1$ or n then one of them will not be present.

First we estimate I_1. For $k = j - i$, $i > 1$, we have

$$\begin{aligned}h_k(x) &= \frac{(1 - x x_k)\, T_n(x)^2}{n^2 (x - x_k)^2} \\ &= \frac{(1 - x^2)\, T_n(x)^2}{n^2 (x - x_k)^2} + \frac{x T_n(x)^2}{n^2 (x - x_k)} \\ &\equiv s_k(x) + t_k(x), \text{ say.}\end{aligned}$$

Now

$$\begin{aligned}s_k(x) &= \frac{\sin^2 \theta \cdot T_n(x)^2}{4n^2 \sin^2 \frac{1}{2}(\theta + \theta_k) \cdot \sin^2 \frac{1}{2}(\theta - \theta_k)}, \qquad x = \cos \theta, \\ &\leqslant \frac{T_n(x)^2}{4n^2 \sin^2 \frac{1}{2}(\theta - \theta_k)} \qquad \text{by Lemma 2a} \\ &= O(1)\, T_n(x)^2\, i^{-2},\end{aligned}$$

and

$$\begin{aligned}t_k(x) &= \frac{x T_n(x)^2}{2n^2 \sin \frac{1}{2}(\theta + \theta_k) \sin \frac{1}{2}(\theta - \theta_k)} \\ &= O(1)\, T_n(x)^2 i^{-2}, \qquad \text{by Lemma 2b.}\end{aligned}$$

Therefore $h_k(x) = O(1)\, T_n(x)^2 i^{-2}$. Then, using Lemma 1 we obtain

$$\begin{aligned}I_1 &= \sum_{i=1}^{j-1} |f(x_i) - f(x)|\, h_i(x) \\ &= O(1)\, T_n(x)^2 \sum_{i=1}^{n} i^{-2} [\omega(f; i(\sin \theta)/n) + \omega(f; i^2/n^2)]\end{aligned}$$

and so, by using Saxena's methods [9],

$$I_1 = O(1)\frac{T_n(x)^2}{n}\sum_{k=1}^{n}[\omega(f;(1-x^2)^{1/2}/k)+\omega(f;1/k^2)]. \tag{3.2}$$

I_3 may be estimated in like manner:

$$I_3 = O(1)\frac{T_n(x)^2}{n}\sum_{k=1}^{n}[\omega(f;(1-x^2)^{1/2}/k)+\omega(f;1/k^2)]. \tag{3.3}$$

It remains to estimate I_2.

$$\begin{aligned} I_2 &= |f(x_j)-f(x)|\, h_j(x) \\ &\leqslant |f(x_j)-f(x)| \\ &\leqslant \omega(f;|\theta-\theta_j|) \\ &\leqslant 2\omega\left(f;\frac{|T_n(x)|}{n}\right) \qquad \text{by Lemma 3.} \end{aligned} \tag{3.4}$$

Formulae (3.1)–(3.4) imply the result stated in Theorem 7.

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